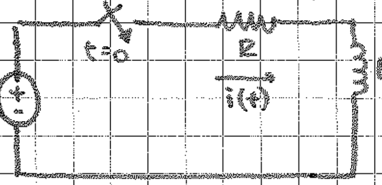
## • The Sinusoidal Response

Consider the following circuit :




where $v_s = V_m \cos(\omega t + \phi)$
Assume that $i(0) = 0$
we want to find $i(t)$, $t \geq 0$.

→ This is the step response of an RL-circuit as we have seen before. The difference is that we have sinusoidal source, $v_s$.

* The governing equation of the circuit is:

$$L \cdot \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

* The solution of this equation is:

$$\boxed{i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi + \theta)\, \underbrace{e^{-(R/L)t}}_{\substack{t \to \infty \\ \text{First term} \to 0}} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)} \;*$$

where $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$

- The first term of this solution is called the "transient response"

- The second term of this solution is called the "steady state response"

- Because for $t \geq 0$, the first term decreases rapidly, and the second term starts to dominate

- So, the first term disappears after a while, and the second term will remains.

- We can make the following conclusions for the steady state response:

1) It is also sinusoidal
* 2) It has the same frequency as the source (due to linearity!)
3) Amplitude and phase changes.
4) Solutions of the diff. eqn is rather long and complicated.

→ The remedy to conclusion 4 is to use "phasors"

- Phasors: are used to make the mathematical solution of the differential eqns for when sinusoidal sources are used

RL
RC
RLC } they are all linear circuit

- The phasor is a complex number that carries the amplitude and phase information of a sinusoidal function.

It is based on the Euler's identity:

$$\boxed{e^{\mp j\theta} = \cos\theta \mp j\sin\theta}$$

Then,

$$\boxed{\cos\theta = Re\{e^{j\theta}\} \text{ and } \sin\theta = Im\{e^{j\theta}\}}$$

Then, we can rewrite the sinusoidal source as:

$v = V_m\cos(\omega t+\theta)$ OR $v = V_m \cdot Re\{e^{j(\omega t+\theta)}\}$ OR $v = V_m Re\{e^{j\omega t} \cdot e^{j\theta}\}$

$$\boxed{v = Re\{V_m e^{j\omega t} e^{j\theta}\}}$$

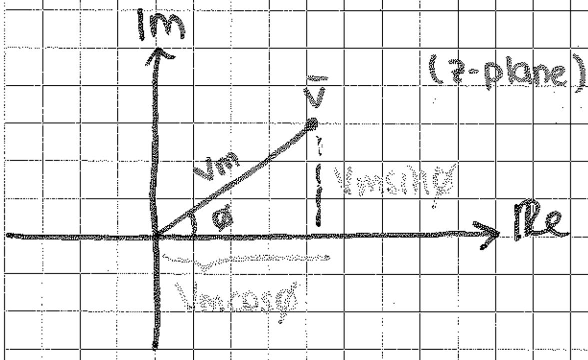
- The term inside the braket that carries the amplitude and phase information is called the "Phasor" of sinusoidal function, $v$.

$$\boxed{\underbrace{\bar{V} = V_m e^{j\phi}}_{\text{Phasor}}} \quad \text{so,} \quad \boxed{\underbrace{\bar{V} = V_m e^{j\theta} = P\{V_m\cos(\omega t+\phi)\}}_{\text{polar form}}}$$

→ In rectangular form,

$$\boxed{\underbrace{\bar{V} = V_m\cos\phi + jV_m\sin\phi}_{\text{rectangular form}}}$$

- We use the following notation: $\boxed{A\angle\phi = Ae^{j\phi}}$



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- **Inverse Phasor Transform**

$$\boxed{P^{-1}\{V_m e^{j\phi}\} = \mathrm{Re}\{V_m e^{j\phi} e^{j\omega t}\}}$$ is the formula of the inverse phasor transform

For the current;

$$i_{ss}(t) = \mathrm{Re}\{I_m e^{j\beta} e^{j\omega t}\}$$

Ex: If $y_1 = 20\cos(\omega t - 30^\circ)$ and $y_2 = 40\cos(\omega t + 60^\circ)$

a) Solve $y = y_1 + y_2$ by using trigonometric identities

b) Solve $y = y_1 + y_2$ by using phasors

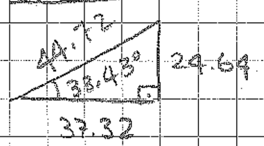
Ans: a) $y_1 = 20\cos(\omega t)\cos 30^\circ + 20\sin(\omega t)\sin 30^\circ$

$$y_2 = 40\cos(\omega t)\cos 60^\circ + 40\sin(\omega t)\sin 60^\circ$$

$$y_1 + y_2 = (20\cos 30 + 40\cos 60)\cos(\omega t) + (20\sin 30 + 40\sin 60)\sin(\omega t)$$

$$= 37.32\cos\omega t - 24.64\sin\omega t$$

We assume;




$$y = 44.72\left(\frac{37.32}{44.72}\cos\omega t - \frac{24.64}{44.72}\sin\omega t\right)$$

$$= 44.72\,(\cos(33.43^\circ)\cos\omega t - \sin(33.43^\circ)\sin\omega t)$$

OR (from $\cos A\cos B - \sin A\sin B = \cos(A+B)$)

$$\boxed{y = 44.72\cos(\omega t + 33.43^\circ)}$$

b) $\bar{Y} = \bar{Y_1} + \bar{Y_2} = 20\angle -30^\circ + 40\angle 60^\circ$

$$\bar{Y} = (17.32 - j\cdot 10) + (20 + j\cdot 34.64)$$

$$= 37.32 + j\cdot 24.64$$

and $= 44.72\angle 33.43^\circ$

$$Z = a + jb = Ae^{j\phi}$$

$$Z = \underbrace{A\cos\phi}_{a} + j\underbrace{\sin\phi}_{b}$$

$$y = \mathrm{Re}\{\bar{Y} e^{j\omega t}\}$$

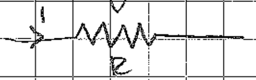
$$= \mathrm{Re}\{44.72 e^{j33.43^\circ} e^{j\omega t}\}$$

$$\boxed{y = 44.72\cos(\omega t + 33.43^\circ)}$$ the same, but much faster and simpler!

(R, C, L)

## • Passive Circuit Elements in Frequency Domain :

### * V-I Relation for a Resistor




$v = R\left[I_m \cos(\omega t+\theta)\right]$
$= R I_m \cos(\omega t+\theta)$

$$\boxed{\bar{V} = R I_m e^{j\theta} = R I_m \angle\theta} \quad \text{OR} \quad \boxed{\bar{V} = R\bar{I}}$$

### * V-I Relation for Inductor

$$v = L\cdot\frac{di}{dt} = -\omega L I_m \sin(\omega t+\theta)$$

OR

$$v = -\omega L I_m \cos(\omega t + \theta - \frac{\pi}{2})$$

and

$$\bar{V} = -\omega L I_m e^{j(\theta-\pi/2)} = -\omega L I_m e^{j\theta}\cdot \underbrace{e^{-j\pi/2}}_{-j} = j\omega I_m e^{j\theta}$$

$$\boxed{\bar{V} = j\omega L\bar{I}}$$

### * V-I Relation for Capacitor

$$i = C\cdot\frac{dv}{dt} \quad \text{where } v = V_m \cos(\omega t+\theta)$$

then, $i = -C V_m \omega \sin(\omega t+\theta)$

$\vdots$ (same steps as inductor)

$$\boxed{\bar{I} = j\omega C\bar{V}} \quad \text{OR} \quad \boxed{\bar{V} = \frac{\bar{I}}{j\omega C} = -j\frac{\bar{I}}{\omega C}}$$

## ⊛ IMPEDENCE & REACTANCE :

Define, $\bar{V} = Z\bar{I}$ (where Z: impedence (frequency dependent resistance))

X = Reactance = Imaginary part of the impedence
R = Resistance = Real part of the impedence

| | Z | X |
|---|---|---|
| Resistor | $R$ | — |
| Inductor | $j\omega L$ | $\omega L$ |
| Capacitor | $\frac{-j}{\omega C}$ | $\frac{-1}{\omega C}$ |

$Y = \frac{1}{Z}$ = admittance ($\mho$ = mho's or S, siemens) , $\frac{1}{X}$ = Susceptance (S)

$Z$ = resistance depends on frequency

# *KIRCHOFF'S LAWS IN FREQUENCY DOMAIN

## • KVL in Phasors

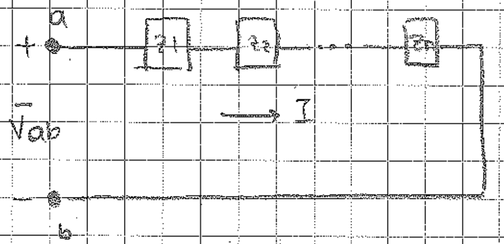
$$\boxed{\bar{V}_1 + \bar{V}_2 + \dots + \bar{V}_n = 0}$$

## • KCL in Phasors

$$\boxed{\bar{I}_1 + \bar{I}_2 + \dots + \bar{I}_n = 0}$$

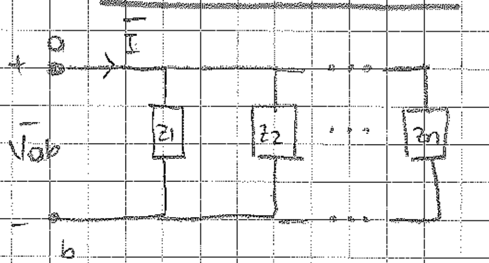
## • Series, Parallel and Δ-Y Simplifications

### 1) Series Connections



$$Z_{ab} = \frac{\bar{V}_{ab}}{\bar{I}} = Z_1 + Z_2 + \dots + Z_n$$

### 2) Parallel Connections



$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \dots + \frac{1}{Z_n}$$

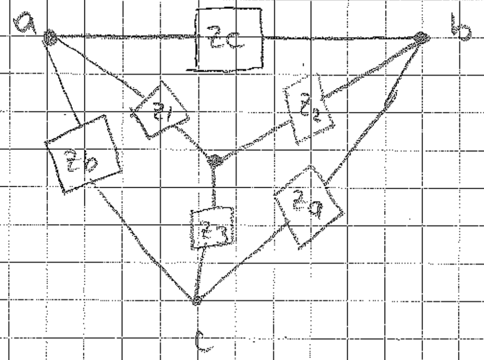
$Y_{ab}$ = Admittance = $G + jB$ (S)

conductance

suspectance

$$\boxed{Y_{ab} = Y_1 + Y_2 + \dots + Y_n}$$

### 3) Δ-Y Transformation



$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}, \quad Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

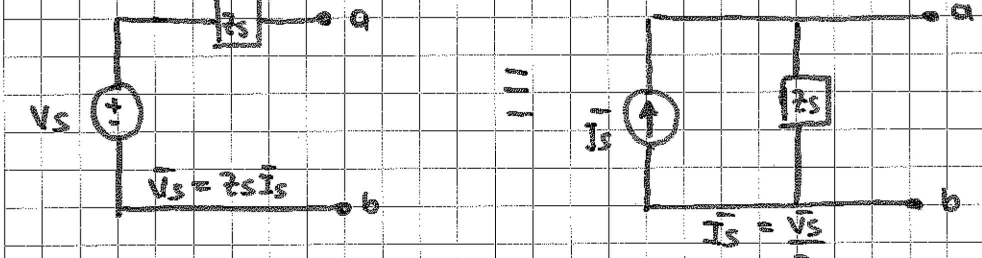
$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3}$$

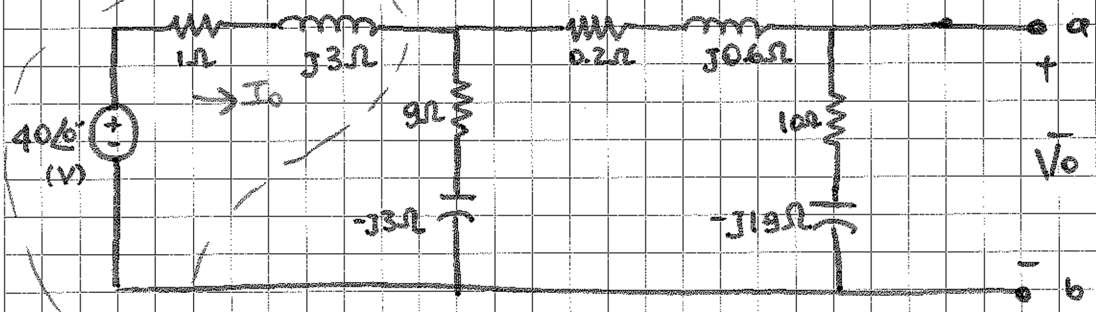
# * SOURCE TRANSFORMATIONS, THÉVENIN & NORTON CIRCUITS

The transformation is as follows:




$\bar{V}_s = Z_s \bar{I}_s$ $\qquad$ $\bar{I}_s = \dfrac{\bar{V}_s}{Z_s}$

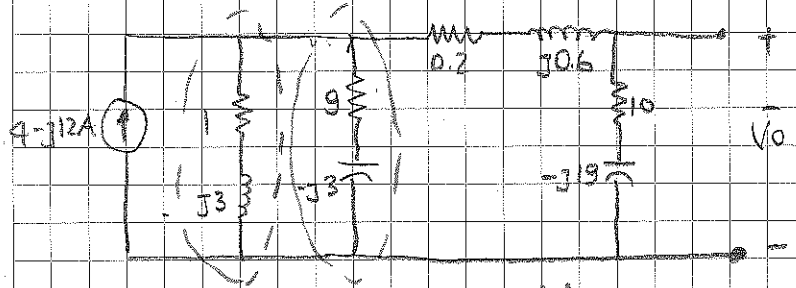
**EX**: Find $\bar{V}_o$ in the circuit below:




Ans: → Replace by Norton Equivalent

$$\bar{I} = \frac{40}{1+j3} \cdot \frac{(1-j3)}{(1-j3)} = \frac{40}{10}(1-3j) = \boxed{4-j12\,(A)}$$

Thus we have;

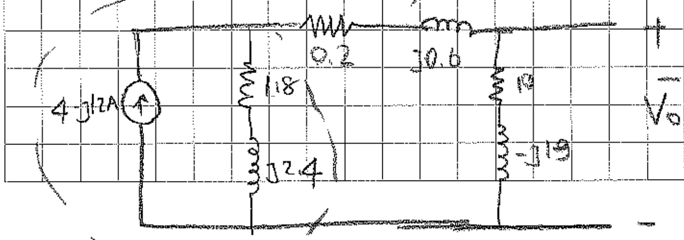



$Z_1 \quad Z_2 \Rightarrow$ parallel

$$\frac{1}{Z_t} = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$\frac{1}{Z_t} = \frac{(1+j3)(9-j3)}{10} = 1.8 + j2.4\,\Omega$$

Now we have;




$$\bar{V} = (4-j12)(1.8+j2.4)$$

$$\boxed{\bar{V} = 36-j12\,V}$$

→ Transform to Thévenin Equivalent